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Noncommutativity and Spontaneously Broken Gauge Theories

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Abstract

We examine the spontaneous symmetry breaking of gauge theories in the framework of noncommutative field theory. We consider a noncommutative $U(2)$ Higgs model with matter in the adjoint representation, that is the simplest example of a noncommutative gauge theory that is both classically well-defined in R_ξ gauges and has an interacting commutative limit. In the ungauged global limit, the quantum theory of this model exhibits the violations of Goldstone's theorem by renormalization effects of the type we had previously found in the noncommutative linear sigma model. In the noncommutative $U(2)$ gauge theory case we also find that spontaneous breaking of the gauge symmetry is in conflict with continuum renormalization of the theory; explicit calculation shows that the physical Higgs (inverse) propagator receives divergent, gauge-dependent, counterterm contributions, even on-shell. Thus we conclude that the noncommutative renormalization of theories with global or gauge symmetries, does not necessarily respect the spontaneous breaking of those symmetries, in contrast to the case of commutative theories where spontaneously broken symmetries can be consistently renormalized.

1 Introduction

Recently field theories on noncommutative spacetime backgrounds have been the subject of intense scrutiny [1]. Part of this motivation stems from the fact that noncommutative $U(N)$ gauge theories arise on D-branes in the presence of a constant NS-NS B-field background, in the zero-slope, field theoretic limit of string theory [2],[3]. Thus it is essential to understand the behaviour of such noncommutative field theories.

The general scheme for defining such field theories with the canonical noncommutative spacetime structure inspired by constant NS-NS B-field backgrounds, and defined by $[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}$, $\theta^{\mu\nu}$ real, constant and antisymmetric, is to invoke Weyl-Moyal correspondence. This has the effect of replacing the underlying noncommutative spacetime with a commutative spacetime at the expense of replacing the ordinary pointwise product of spacetime dependent functions with an infinitely nonlocal star product. The induced momentum space Feynman rules for interaction vertices associated with a given field theory then involve momentum-dependent phases, which generically split a graph (at least at one-loop) into planar and nonplanar parts. The former are identical to the usual commutative graphs (up to a total phase depending only on the external momenta, and a combinatorial reweighting), and in particular possess the usual divergence structure associated with a commutative quantum field theory. The latter, nonplanar components are explicitly finite (at least at one-loop) because of oscillatory damping due to the phases, and replace an ultraviolet divergence with an infrared divergence in the external momenta [4].

Superficially, as a consequence of the finiteness of nonplanar graphs, and of the similar divergence structure of the planar graphs, one might conclude that the renormalization of noncommutative field theories proceeds as in the commutative theory, because the counterterm structure is formally the same. However, as is well-known, the renormalization of spontaneously broken theories, with either underlying global or gauge symmetries, is more subtle because the number of counterterm vertices exceeds the number of renormalization parameters. As a result, the renormalizability of (commutative) spontaneously broken theories hinges in general on intricate graphwise cancellations [5], [6] order by order in perturbation theory. Thus it is of obvious interest to examine whether or not these cancellations persist in noncommutative field theories.

In a previous paper [7] we studied the spontaneous symmetry breaking of a global symmetry in the noncommutative deformation of the linear sigma model. We found that one-point tadpoles of the sigma at one-loop were insensitive to the noncommutativity because no external momentum flows into the trilinear tadpole vertex. Thus the one-point sigma counterterm is identical to the one in the commutative limit, which in turn fixes the pion mass counterterm to be the same as its commutative limit. On the other hand, the planar components of the 1PI graphs contributing to the one-loop pion (inverse) propagator renormalization are reweighted with respect to the corresponding commutative graphs. As a consequence, there is an unavoidable UV cutoff dependence (for nonzero external momentum) *after* renormaliza-

tion, signalling the nonexistence of a continuum limit, and noncommuting UV ($\Lambda_{UV} \rightarrow \infty$) and IR ($p \rightarrow 0$) limits.

When we gauge a model with spontaneous symmetry breaking, the Goldstone modes (the pions of the linear sigma model) become the longitudinal modes of the corresponding (now massive) gauge fields. Thus, it is of interest to understand how noncommutativity affects spontaneously broken gauge theories; it is our intention to undertake such an analysis in this paper.

The simplest commutative model of spontaneous symmetry breaking of a gauge symmetry is the Abelian Higgs model, so it would seem natural that we investigate its noncommutative deformation. Unfortunately, as we will show in the next section, there are difficulties in defining a consistent noncommutative Higgs model based on the NC $U(1)$ group, with the scalar in the fundamental representation, outside of complete gauge fixing to the unitary gauge. This is fundamentally due to the fact that while commutative $U(1)$ and commutative $SO(2)$ are isomorphic, the noncommutative generalizations of these gauge groups are not, and in particular the noncommutative $SO(N)$ gauge algebra [8] does not close.¹ Since the usual treatment of the Abelian Higgs model, involves implicit use of this trivial commutative isomorphism in order to select a vacuum state, we expect that problems will arise in defining the model's noncommutative deformation.

The problems that arise in defining classical spontaneous symmetry breaking in the fundamental representation of NC $U(1)$ disappear if we work at the outset in a self-conjugate representation of the gauge group, such as the adjoint representation, and leave off-diagonal elements in gauge and scalar field multiplets complex. Now while the adjoint representation of the noncommutative $U(1)$ group is nontrivial, since its classical commutative limit is trivial, we will instead study the noncommutative $U(2)$ model, with scalars Φ in the adjoint representation. Within this model we find interesting results already in the global (ungauged) limit. In particular we find our previous [7] results are partially ameliorated, and Goldstone's theorem is partially restored. The remaining post-renormalization divergence of the one-loop inverse propagator corrections to the now-complex Goldstone mode is dependent only on the coupling to the $\text{Tr}(\Phi^2)^2$ term in the scalar potential. The piece dependent on the coupling to $\text{Tr}(\Phi^4)$ is surprisingly cancelled by a purely noncommutative graph involving a coupling of the $U(1)$ component of the scalar field multiplet to the Goldstone mode.

In the full gauge theory we will examine the simplest physical quantity which must be gauge independent: the on-shell mass renormalization of the physical Higgs particle. Guided by the careful treatment of the ordinary Abelian Higgs model at one-loop provided by Appelquist et al. [9], and by our previous observation that one-point tadpoles do not see the noncommutativity at one-loop we demonstrate that in this model the Higgs acquires a gauge-dependent mass renormalization, even when evaluated on the mass shell. Interestingly, the residual gauge dependence again explicitly depends only on the coupling to the $\text{Tr}(\Phi^2)^2$ term.

¹Note this is strictly a restriction on allowed gauge groups, and not global symmetry groups which do not see the star product.

2 Issues in defining the NC Abelian Higgs Model

In this section we will exhibit the difficulties that arise already at the classical level when trying to construct a noncommutative deformation of the ordinary Abelian Higgs Model (AHM) outside of complete gauge fixing to the unitary gauge.

The commutative AHM is specified classically by the Lagrangian density for a complex scalar ϕ , and a Abelian gauge field A^μ given by

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + (D_\mu\phi)^\dagger D^\mu\phi + \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 \quad (1)$$

where as usual $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $D_\mu\phi = (\partial_\mu - igA_\mu)\phi$, g is the gauge coupling, and λ is the scalar self-coupling. It is invariant under the local $U(1)$ gauge transformations

$$\phi \rightarrow e^{i\alpha(x)}\phi \quad , \quad A_\mu \rightarrow A_\mu + \frac{1}{g}\partial_\mu\alpha \quad (2)$$

If $\mu^2 > 0$, the scalar potential is minimized for the translationally invariant vacuum

$$|\phi|^2 = \frac{\mu^2}{2\lambda} \equiv \frac{v^2}{2} \quad (3)$$

and the symmetry is spontaneously broken by this vacuum expectation value (VEV). If we write the complex scalar in terms of real components as

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \quad (4)$$

we may use the gauge freedom to rotate the VEV into the real component of the field ϕ_1 , and then define shifted fields,

$$\sigma = \phi_1 - v \quad , \quad \pi = \phi_2 \quad (5)$$

which now have zero VEVs, and where we have chosen to use sigma-model language for future convenience. However the existence of the *unitary* gauge, reveals that the imaginary component is unphysical as is most easily seen by writing the original complex field in polar coordinates, shifting only the modulus

$$\phi(x) = \frac{1}{\sqrt{2}}e^{\frac{i\xi(x)}{v}}(h(x) + v) \quad (6)$$

and then selecting the gauge function $\alpha(x) = -i\xi(x)/v$, which manifestly transforms away the ξ field (assuming of course that we write the Lagrangian in terms of the gauge-transformed gauge field). Note that this gauge exists only in the case where the symmetry is spontaneously broken, wherein $v \neq 0$. For small oscillations about the minimum, expanding the exponential reveals that h and ξ have the same particle interpretation (for say canonical quantization) as σ and π respectively [10], so in effect we have transformed away the imaginary component of the complex scalar.

Now consider the putative noncommutative deformation of this theory via the star product defined by

$$f(x) * g(x) = \exp(i\theta^{\mu\nu} \partial_\mu^y \partial_\nu^z) f(y) g(z) |_{y,z \rightarrow x} \quad (7)$$

which satisfies the involution relation

$$(f(x) * g(x))^\dagger = g(x)^\dagger * f(x)^\dagger \quad (8)$$

as a consequence of the antisymmetry of θ . In our subsequent discussions of spontaneous symmetry breaking, we will search only for translationally invariant vacuum states, which do not see the star product. Noncommutative $U(1)$ gauge transformations are defined by the power series,

$$e_*^{i\lambda(x)} \equiv 1 + i\lambda(x) - \frac{1}{2!}\lambda(x) * \lambda(x) + \dots \quad (9)$$

(noncommutative $U(N)$ gauge transformations are similarly defined). Thus the action of the noncommutative $U(1)$ group on a complex scalar $\phi(x)$ (in the fundamental representation) is simply

$$\phi(x) \rightarrow e_*^{i\lambda(x)} * \phi(x) \equiv U_1(x) * \phi(x) \quad (10)$$

Using the involution relation (8), the antifundamental representation is given by

$$\phi^\dagger \rightarrow \phi^\dagger(x) * e_*^{-i\lambda(x)} \equiv \phi^\dagger(x) * U_1^\dagger(x) \quad (11)$$

We also note that noncommutative $U(1)$ admits a nontrivial adjoint representation

$$\Phi \rightarrow U_1(x) * \Phi(x) * U_1^\dagger(x) \quad (12)$$

that becomes trivial in the commutative limit, where we may commute Φ through U_1^\dagger . In the commutative case we may equivalently consider the complex scalar to be two real scalars, and write $U(1)$ gauge transformations as $SO(2)$ gauge transformations. However, this decomposition fails in the noncommutative case because the left and right multiplication by the $U(1)$ gauge transformations represent distinct orderings. The gauge transformation (10) of $\phi(x) = 1/\sqrt{2}(\phi_1(x) + i\phi_2(x))$ expanded to first order in the gauge parameter implies

$$\phi_1(x) \rightarrow \phi_1(x) - \lambda(x) * \phi_2(x) \ , \ \phi_2(x) \rightarrow \phi_2(x) + \lambda(x) * \phi_1(x) \quad (13)$$

whereas the gauge transformation (11) of $\phi^\dagger = 1/\sqrt{2}(\phi_1(x) - i\phi_2(x))$ implies

$$\phi_1(x) \rightarrow \phi_1(x) - \phi_2(x) * \lambda(x) \ , \ \phi_2(x) \rightarrow \phi_2(x) + \phi_1(x) * \lambda(x) \quad (14)$$

Thus the two representations imply different gauge transformation laws for the real and imaginary components of the complex scalar. Since gauge-invariant terms in the Lagrangian are built by coupling fundamental to antifundamental representations, this decomposition does not respect the gauge symmetry. We will henceforth refer to a complex field in a representation of a gauge group that does not allow such decompositions, as *intrinsically* complex. Obviously in the global symmetry limit, or the commutative limit this problem does not arise since the ordering of the gauge parameter with respect to a field is immaterial. We also note that this is a direct consequence of the involution relation (8). In fact, as

Matsubara demonstrated [8], as a direct consequence of this relation, the gauge algebras for NC $SO(N)$ do not close under the Moyal bracket, whereas those for NC $U(N)$ do close². We regard the above observation as a manifestation of this result.

What is the relevance of this to the AHM? There is no problem with the noncommutative deformation of (1) in the symmetric phase, as long as we leave the field ϕ complex. The problem only arises in the case where the symmetry is spontaneously broken, and we have to arbitrarily select a vacuum state in order to define a perturbation theory about a stable vacuum, which entails a decomposition, and field redefinition, of the kind above. To illustrate the difficulty of evading this result, suppose we not make the decomposition explicit, but instead try to write the Lagrangian in terms of $\phi + \phi^\dagger$ (the physical Higgs up to a constant), and the orthogonal combination $i(\phi^\dagger - \phi)$. Now the problem arises in trying to construct a gauge-invariant scalar potential written in terms of the degrees of freedom relevant to a spontaneously broken model. As Aref'ava et. al [11] originally observed, while the noncommutative ordering

$$\phi^\dagger * \phi * \phi^\dagger * \phi \quad (15)$$

possesses noncommutative $U(1)$ gauge invariance, the other possible field ordering (modulo cyclic permutation)

$$\phi^\dagger * \phi^\dagger * \phi * \phi \quad (16)$$

does not. But products of $\phi + \phi^\dagger$ (in particular $(\phi + \phi^\dagger)^4$) necessarily invoke both orderings, so the scalar potential cannot be written in a gauge-invariant way in terms of these degrees of freedom. Furthermore, the infinitesimal gauge transformations for these combinations read

$$\delta_\lambda(\phi + \phi^\dagger) = i(\lambda * \phi - \phi^\dagger * \lambda) \ , \ \delta_\lambda i(\phi^\dagger - \phi) = \phi^\dagger * \lambda + \lambda * \phi \quad (17)$$

which shows explicitly the failure of the gauge transformations to close with respect to these degrees of freedom.

Finally, we might consider representing the complex field in polar form, as in the discussion of the unitary gauge above. This introduces a star product into the definition of a single complex field via

$$\phi(x) = \frac{1}{\sqrt{2}} e_*^{i\xi(x)/v} * [h(x) + v] \equiv \text{Unitary} \times \text{Hermitian} \quad (18)$$

Again, for small oscillations this reduces to the usual decomposition, and the linear order component fields do not see the star product. Acting on the left with a NC $U(1)$ gauge transformation, with gauge parameter $\alpha(x) = -\xi(x)/v$ removes the phase as in the commutative case, and so still defines a unitary gauge. The remaining field is hermitian, and performing the corresponding gauge transformation on ϕ^\dagger yields no contradiction. Only for this particular gauge choice is there no problem: the gauge transformation is exactly such that, order by order in the gauge parameter, all terms depending on ξ are eliminated, and there is no contradiction between the gauge transformation of ϕ and that of ϕ^\dagger .

²In fact Seiberg and Witten [3] originally noted that it was not obvious how gauge groups other than $U(N)$ could be obtained in this framework.

In light of the discussion above, it is not clear what the meaning of the theory is in the spontaneously broken phase outside of the complete gauge-fixing to the unitary gauge. Since we are ultimately interested in gauge (in)dependence at the quantum level, working in a completely fixed gauge sheds no light on this question. Furthermore, the unitary gauge should really be viewed as the limit of a class of renormalizable gauges (the R_ξ gauges) where the gauge independence of Green's functions (evaluated on-shell) is established a priori. In light of our eventual results, assuming the existence of the unitary gauge *a priori* is suspect. Finally, even in the commutative case, the unitary gauge is a nonrenormalizable gauge, and a finite S-matrix results only from cancellations among divergent Green's functions, which are assured to occur precisely because the unitary gauge represents a limit of the renormalizable R_ξ gauges, as the gauge parameter is taken to infinity. Therefore as a consequence of these issues, we will not pursue the noncommutative generalization of the AHM (which possesses an intrinsically complex field) any further, instead focusing on the deformations of more complex models.

The above problems are avoided by working in a self-conjugate representation such as the adjoint representation. Furthermore, as we will see in the next sections, working in the adjoint representation for noncommutative theories is natural because the expansion of terms in the scalar potential or in the matter covariant derivative in component fields automatically captures all noncommutative orderings of a given particular commutative term. While the off-diagonal components of the noncommutative matter field multiplets in the adjoint are intrinsically complex, and should not be decomposed, it is the diagonal components which determine the symmetry breaking pattern in adjoint representations, and they are of course explicitly real; thus the study of spontaneous symmetry breaking of noncommutative gauge symmetries with matter in the adjoint is possible, at least classically.

As mentioned above, the adjoint of NC $U(1)$ is in fact a nontrivial representation that has a trivial commutative limit, since commutative $U(1)$ is Abelian, wherein the gauge field decouples from the real scalar. For this reason, we will not study this model, instead choosing to study a spontaneously broken NC $U(2)$ model, with scalars in the adjoint representation for the remainder of this paper. This represents a minimal model whose spontaneously broken phase is well-defined classically with respect to gauge invariance, and which has an interacting commutative limit.

3 NC $U(2)$ Model: Global Theory of SSB

Before studying the full gauge theory, and in light of our results in [7], we will first examine the status of Goldstone's theorem in the global model with scalars in the adjoint representation of $U(2)$.

We write the scalars in the adjoint of $U(2)$ as

$$\Phi = \phi_a T^a = \frac{1}{2} \begin{pmatrix} \phi_4 + \phi_3 & \sqrt{2}\phi^* \\ \sqrt{2}\phi & \phi_4 - \phi_3 \end{pmatrix} \quad (19)$$

where T^a are the canonical generators of $U(2)$: $T^a = \sigma^a/2$, for $a = 1, 2, 3$ and $T^4 = I_2/2$. We leave the off-diagonal elements complex, and with respect to components of field multiplets we use \dagger and $*$ interchangeably, where they coincide. The global $U(2)$ symmetry transformation acts as

$$\Phi \rightarrow U\Phi U^\dagger \quad (20)$$

and does not involve the star product because the symmetry is global at this point. The Lagrangian density for the global model we consider is defined by³

$$\mathcal{L} = \text{Tr}(\partial_\mu \Phi * \partial^\mu \Phi) + \mu^2 \text{Tr}(\Phi * \Phi) - \lambda_1 \text{Tr}(\Phi * \Phi * \Phi * \Phi) - \lambda_2 (\text{Tr}(\Phi * \Phi))^2 \quad (21)$$

since the adjoint representation admits two invariant quartic terms. For notational brevity all star products will be suppressed henceforth, unless there is danger of confusion. Furthermore, we will implicitly use the identity

$$\int A_1 * \dots * A_n = \int A_{\sigma(1)} * \dots * A_{\sigma(n)} \quad (22)$$

(where $\{\sigma(1) \dots \sigma(n)\}$ represents any cyclic permutation of $\{1 \dots n\}$), with the understanding that all Lagrangian density terms sit under a spacetime integral. This identity means that quadratic terms in the action are identical to their commutative counterparts.

Let us now consider spontaneous symmetry breaking which occurs for $\mu^2 > 0$ (we take $\lambda_i > 0$). Throughout the remainder of this paper, we will consider only translationally invariant vacua⁴. Then Φ acquires a vacuum expectation value, say Φ_0 , and since it is a Hermitian (but not necessarily traceless) matrix, we analyze it by diagonalization to the form

$$\Phi_0 = \frac{1}{2} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad (23)$$

whence the potential becomes

$$V(a, b) = -\frac{\mu^2}{4}(a^2 + b^2) + \frac{\lambda_1}{16}(a^4 + b^4) + \frac{\lambda_2}{16}(a^4 + 2a^2b^2 + b^4) \quad (24)$$

This is minimized for

$$a^2 = b^2 = \frac{\mu^2}{\frac{\lambda_1}{2} + \lambda_2} \equiv \frac{\mu^2}{\lambda} \quad (25)$$

³While the most general renormalizable scalar potential for a $U(2)$ model includes trace invariants of products of an *odd* number of Φ s, they cannot affect any of our subsequent results as we will point out in the Discussion.

⁴As Gubser and Sondhi have argued [12], more exotic vacua such as stripe phases are possible in non-commutative theories.

The states corresponding to $a = b$, which are degenerate in energy with the states corresponding to $a = -b$, and admitted because we are considering $U(2)$ and not simply $SU(2)$, do not reflect spontaneously broken states, because Φ_0 is then proportional to the identity and so manifestly commutes with all of the generators. They correspond to unimportant constant shifts in the $U(1)$ component ϕ_4 , so we pay no further attention to them (see Discussion). On the other hand, the states corresponding to $a = -b$ do yield spontaneously broken vacua, since they do not commute with the T^1 and T^2 generators and reflect a vacuum expectation value for the field ϕ_3 .

In notation suggestive of the linear sigma model, we expand around the vacuum $b = -a < 0$ (without any loss of generality), defining σ and π through

$$\Phi' = \frac{1}{2} \begin{pmatrix} \phi_4 + \sigma & \sqrt{2}\pi^* \\ \sqrt{2}\pi & \phi_4 - \sigma \end{pmatrix} \equiv \Phi - \Phi_0 \quad (26)$$

so that $\phi_3 = \sigma + a$. Expanding the scalar potential in terms of these variables yields

$$\begin{aligned} V = & \frac{1}{2}(2\mu^2)\sigma^2 + \frac{1}{2}(\lambda_1 a^2)\phi_4^2 + \frac{\lambda_1 + \lambda_2}{2}\pi^*\pi\pi^*\pi + \frac{\lambda_2}{2}\pi^*\pi^*\pi\pi + \frac{\lambda_1 + \lambda_2}{2}(\pi^*\pi + \pi\pi^*)\sigma^2 \\ & - \frac{\lambda_1}{2}\pi^*\sigma\pi\sigma + \frac{\lambda_1 + \lambda_2}{2}(\pi^*\pi + \pi\pi^*)\phi_4^2 + \frac{\lambda_1}{2}\pi^*\phi_4\pi\phi_4 + a\lambda(\pi^*\pi + \pi\pi^*)\sigma \\ & + \frac{\lambda}{4}(\sigma^4 + \phi_4^4) + \lambda a\sigma^3 + \frac{\lambda_1 + \lambda_2}{2}\sigma^2\phi_4^2 + \frac{\lambda_1}{4}\sigma\phi_4\sigma\phi_4 + (\lambda + \lambda_1)a\phi_4^2\sigma \\ & + \frac{\lambda_1}{2}[\pi^*\phi_4\pi\sigma - \pi^*\sigma\pi\phi_4 + a(\pi\pi^*\phi_4 - \pi^*\pi\phi_4)] \end{aligned} \quad (27)$$

using $\lambda = \lambda_1/2 + \lambda_2$. We note that working in the adjoint representation has automatically induced all possible noncommutative orderings, and has induced two purely noncommutative interactions that will yield interesting results in the following.

The symmetrized Feynman rules are listed in the Appendix for the full NC $U(2)$ gauge theory. The global theory is obtained by ignoring all gauge interactions ($g = 0$), and dropping the R_ξ -gauge induced pion mass. In light of our results concerning Goldstone's theorem in the noncommutative linear sigma model reported in [7], we re-examine the issue in this model. To simplify the discussion relative to that occurring in [7], we will not *a priori* impose the vanishing of the tadpole as a renormalization condition. Instead we will include the one-point tadpole contributions, and their counterterm directly in calculating the mass renormalization of the pion. In this completely equivalent language, the two counterterms present cancel each other, up to the wavefunction renormalization, so the sum of the one-particle irreducible (1PI) graphs and the one-point tadpole insertions must be automatically finite up to wavefunction renormalization (and for Goldstone's theorem to hold at one-loop, must vanish in the $p \rightarrow 0$ limit). Furthermore, to exhibit the essentially algebraic nature of the result, we will expand the non-phase part of the integrands about zero-external momentum, in the cases where there are two propagators in the loop using the Taylor expansion

$$\frac{1}{k^2[(p+k)^2 - m^2]} = \frac{1}{k^2(k^2 - m^2)} - p_\mu \frac{2k^\mu}{k^2[k^2 - m^2]^2} + \dots \quad (28)$$

and then note that the p -dependent terms yield finite loop-momentum integrals (for all p), and vanish as $p \rightarrow 0$. Then define the momentum integrals

$$I(m^2) = \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 - m^2}, \quad I_{\theta,p}(m^2) = \int \frac{d^4 k}{(2\pi)^4} \frac{\cos(k \times p)}{k^2 - m^2} = \int \frac{d^4 k}{(2\pi)^4} \frac{e^{ik \times p}}{k^2 - m^2} \quad (29)$$

where $k \times p = k_\mu \theta^{\mu\nu} p_\nu$. In the following solid lines denote the σ , dots denote the ϕ_4 , and dashes denote the π .

Excluding the purely noncommutative interactions for separate consideration, there are four 1PI graphs contributing to the mass renormalization of the complex pion (Goldstone mode) in this model:

$$\begin{aligned} \text{Diagram 1: } \pi \text{ (dashed) } \rightarrow \text{ vertex } \rightarrow \pi^* \text{ (dashed)} \text{ with a dashed loop } &= -2i(\lambda_1 + \lambda_2) \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2} - 2i\lambda_2 \int \frac{d^4 k}{(2\pi)^4} \frac{i \cos^2(\frac{k \times p}{2})}{k^2} \\ &= (2\lambda_1 + 3\lambda_2)I(0) + \lambda_2 I_{\theta,p}(0) \end{aligned} \quad (30)$$

$$\text{Diagram 2: } \pi \text{ (dashed) } \rightarrow \text{ vertex } \rightarrow \pi^* \text{ (dashed)} \text{ with a solid loop } = (\lambda_1 + \lambda_2)I(2\mu^2) - \frac{\lambda_1}{2}I_{\theta,p}(2\mu^2) \quad (31)$$

$$\text{Diagram 3: } \pi \text{ (dashed) } \rightarrow \text{ vertex } \rightarrow \pi^* \text{ (dashed)} \text{ with a dotted loop } = (\lambda_1 + \lambda_2)I(\lambda_1 a^2) + \frac{\lambda_1}{2}I_{\theta,p}(\lambda_1 a^2) \quad (32)$$

$$\begin{aligned} \text{Diagram 4: } \pi \text{ (dashed) } \rightarrow \text{ vertex } \rightarrow \pi^* \text{ (dashed)} \text{ with a solid loop } &= (-2i\lambda a)^2 \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2} \frac{i}{(p+k)^2 - 2\mu^2} \cos^2(\frac{k \times p}{2}) \\ &= 4\lambda^2 a^2 \int \frac{d^4 k}{(2\pi)^4} \frac{1}{2\mu^2} \left[\frac{1}{k^2 - 2\mu^2} - \frac{1}{k^2} \right] \cos^2(\frac{k \times p}{2}) + C^\mu(p)p_\mu \\ &= \lambda \left[I(2\mu^2) - I(0) \right] + \lambda \left[I_{\theta,p}(2\mu^2) - I_{\theta,p}(0) \right] + C^\mu(p)p_\mu \end{aligned} \quad (33)$$

where as discussed above, C^μ is finite for all p . Note, only the second and third graphs have symmetry factors of $1/2$ because the pion is complex.

As discussed in [7], one-loop, one-point tadpoles do not see the noncommutativity because no external momentum can flow into the trilinear vertex occurring in such graphs, so the noncommutative phase always degenerates to one. This is the origin of the post-renormalization UV cutoff dependence at nonzero external momentum in the noncommutative linear sigma model. For our model, these one-point tadpole contributions are given by

$$\begin{array}{c} \text{---}\pi \\ \text{---}\blacktriangleleft \\ p \end{array} \begin{array}{c} \text{---}\text{---}\text{---} \\ \text{---}\bullet \\ \text{---}\text{---}\text{---} \end{array} \begin{array}{c} \text{---}\blacktriangleright \\ p \\ \text{---}\pi^* \end{array} = (-2i\lambda a)^2 \frac{i}{-2\mu^2} iI(0) = -2\lambda I(0) \quad (34)$$

$$\begin{array}{c} \circ \\ | \\ \pi \text{---} \blacktriangleright_p \text{---} \bullet \text{---} \blacktriangleright_p \text{---} \pi^* \end{array} = (-2i\lambda a) \frac{i}{-2\mu^2} (-6i\lambda a) \frac{i}{2} I(2\mu^2) = -3\lambda I(2\mu^2) \quad (35)$$

$$\begin{aligned}
\text{---}\pi \text{---} \text{---} p \text{---} \text{---} \text{---} p \text{---} \text{---} \pi^* &= (-2i\lambda a) \frac{i}{-2\mu^2} (-2i(\lambda + \lambda_1)a) \frac{i}{2} I(\lambda_1 a^2) \\
&= -(\lambda + \lambda_1) I(\lambda_1 a^2)
\end{aligned} \tag{36}$$

Again, only the latter two graphs have symmetry factors of $1/2$, because the pion is complex.

The sum of these seven graphs is given by

$$\Sigma = \frac{\lambda_1}{2} [I(0) - I_{\theta,p}(0)] - \lambda_2 [I(2\mu^2) - I_{\theta,p}(2\mu^2)] - \frac{\lambda_1}{2} [I(\lambda_1 a^2) - I_{\theta,p}(\lambda_1 a^2)] + C^\mu(p) p_\mu \quad (37)$$

In the commutative limit $\theta \rightarrow 0$, this degenerates to the finite term $C^\mu(p)p_\mu$ (which itself vanishes as $p \rightarrow 0$), so the mass counterterm vanishes and this is a demonstration of Goldstone's theorem for this model. However for nonzero θ , the $I(m^2)$ terms are divergent and require regularization, say by a ultraviolet cutoff Λ . But there is no counterterm freedom to cancel the Λ dependence, so for nonzero p and nonzero θ we cannot take the continuum limit; that is, UV ($\Lambda \rightarrow \infty$) and IR ($p \rightarrow 0$) limits do not commute.

So far the discussion parallels that in [7]. However, we have (intentionally) neglected a purely noncommutative graph due to the last interaction in (27) that is present because we are in the adjoint representation. The purely noncommutative interaction generated by $(\pi\pi^* - \pi^*\pi)\phi_4$ yields a graphical contribution given by

$$\begin{array}{c}
\text{\scriptsize π} \quad \text{\scriptsize p} \quad \text{\scriptsize k} \quad \text{\scriptsize π^*} \\
\text{\scriptsize $k+p$}
\end{array}
= (-\lambda_1 a)^2 i^2 \int \frac{d^4 k}{(2\pi)^4} \frac{\sin(\frac{p \times k}{2}) \sin(\frac{-k \times -p}{2})}{k^2 [(p+k)^2 - \lambda_1 a^2]}$$

$$\begin{aligned}
&= \lambda_1^2 a^2 \int \frac{d^4 k}{(2\pi)^4} \frac{\sin^2(\frac{k \times p}{2})}{\lambda_1 a^2} \left[\frac{1}{k^2 - \lambda_1 a^2} - \frac{1}{k^2} \right] + D_\theta^\mu(p) p_\mu \\
&= \frac{\lambda_1}{2} \int \frac{d^4 k}{(2\pi)^4} [1 - \cos(k \times p)] \left[\frac{1}{k^2 - \lambda_1 a^2} - \frac{1}{k^2} \right] + D_\theta^\mu(p) p_\mu \\
&= \frac{\lambda_1}{2} [I(\lambda_1 a^2) - I_{\theta,p}(\lambda_1 a^2)] - \frac{\lambda_1}{2} [I(0) - I_{\theta,p}(0)] + D_\theta^\mu(p) p_\mu \quad (38)
\end{aligned}$$

where again D_θ^μ is finite for all p , and vanishes also in the limit $\theta \rightarrow 0$. Rather unexpectedly, this graph, which manifestly vanishes in the commutative limit, and involves the $U(1)$ component of the matter field, cancels the λ_1 pieces in (37), leaving behind a residual divergence (for nonzero p) that depends only on the coupling to the $\text{Tr}(\Phi^2)^2$ term in the potential.

Thus if we tune the tree-level coupling λ_2 to zero, then at one-loop we recover Goldstone's theorem as a result of this purely noncommutative coupling to the $U(1)$ component field, ϕ_4 . However, we expect to generate the $\text{Tr}(\Phi^2)^2$ interactions radiatively, so we have really just deferred violations of Goldstone's theorem into the two-loop order. As a final note, we observe that the purely noncommutative interactions responsible for this remarkable cancellation do not in fact radiatively generate dangerous $\sigma - \phi_4$ amplitudes (at least at one-loop) as they might superficially appear to. This is essentially because their Feynman rules involve sines (the trademark of an interaction that vanishes in the commutative limit) in such a way that one-loop contributions to such an amplitude vanish as a trivial check reveals. We now turn to the full gauge theory.

4 NC $U(2)$ Model: Gauge Theory of SSB

In this section we will study the spontaneously broken noncommutative $U(2)$ gauge theory with scalars in the adjoint. We will define the R_ξ gauge fixing for the theory so that we may attempt to quantize it; our ultimate interest lying in the study of the gauge (in)dependence of the theory, at the one-loop quantum level. In particular, we will study the simplest quantity which must be gauge-independent: the on-shell mass renormalization of the physical Higgs particle (the σ of our model). Despite the fact that we are working with a noncommutative, and non-Abelian model, our analysis will closely parallel the discussion in the original treatment of the ordinary Abelian Higgs model at one-loop performed by Appelquist et. al. [9]. Such parallels exist because we are studying quantum corrections to the (inverse) Higgs propagator, where the non-Abelian nature of the theory does not play an important role at one-loop.

An important difference with respect to the global theory as treated in [7], is that we do not impose a renormalization condition on the one-point Higgs/sigma tadpoles. This is because the tadpoles themselves are both gauge-dependent, and divergent in the gauge theory, but play an important role in the cancellation of other gauge-dependent contributions to physical renormalization effects [9]. Instead, as in the previous section, we will directly include one-

point tadpole insertions, and their counterterm, in addition to the usual 1PI contributions when calculating the mass-renormalization of the Higgs.

The NC $U(2)$ gauge transformations on the scalar field Φ in the adjoint representation read

$$\Phi \rightarrow U_2 * \Phi * U_2^\dagger \quad (39)$$

To illustrate why no problem occurs in this representation analogous to the problems we encountered earlier when decomposing the complex scalar in the fundamental of NC $U(1)$, expand these gauge transformations to linear order in the gauge parameter

$$\lambda_a T^a = \frac{1}{2} \begin{pmatrix} \lambda_4 + \lambda_3 & \sqrt{2}\lambda^* \\ \sqrt{2}\lambda & \lambda_4 - \lambda_3 \end{pmatrix} \quad (40)$$

whence the components of Φ defined in (19) have infinitesimal transformations

$$\delta\phi \sim \frac{i}{4} \left[\{\lambda, \phi_3\} + [\lambda, \phi_4] + [\lambda_4, \phi] - \{\lambda_3, \phi\} \right] \quad (41)$$

$$\delta\phi^* \sim \frac{i}{4} \left[[\lambda^*, \phi_4] - \{\lambda^*, \phi_3\} + \{\lambda_3, \phi^*\} + [\lambda_4, \phi^*] \right] \quad (42)$$

$$\delta\phi_3 \sim \frac{i}{4} \left[[\lambda_3, \phi_4] + [\lambda_4, \phi_3] + \{\lambda^*, \phi\} - \{\phi^*, \lambda\} \right] \quad (43)$$

$$\delta\phi_4 \sim \frac{i}{4} \left[[\lambda_3, \phi_3] + [\lambda_4, \phi_4] + [\lambda^*, \phi] + [\lambda, \phi^*] \right] \quad (44)$$

$$(45)$$

where λ_i are not to be confused with the scalar potential coupling constants; they will never appear in the same discussion. By repeatedly using (8), it is a trivial exercise to show that these transformations have the required reality properties. This is a direct consequence of working in the adjoint representation. In particular, we have real field components on the diagonal with consistent gauge transformations, which allows us to consider spontaneous symmetry breaking (the pattern of which is determined by the diagonal field components in the adjoint representation). Finally we note the decoupling of the $U(1)$ components of the gauge parameter, and scalar field multiplet in the commutative limit, as required of the adjoint representation.

We now introduce gauge fields in the adjoint of NC $U(2)$ as

$$\mathcal{A}^\mu = \frac{1}{2} \begin{pmatrix} A_4^\mu + A_3^\mu & \sqrt{2}A^{\mu*} \\ \sqrt{2}A^\mu & A_4^\mu - A_3^\mu \end{pmatrix} \quad (46)$$

keeping off-diagonal gauge fields complex. The gauge transformation of \mathcal{A}_μ is given by

$$\mathcal{A}_\mu \rightarrow U_2 * \mathcal{A}_\mu * U_2^\dagger - \frac{i}{g} (\partial_\mu U_2) * U_2^\dagger \quad (47)$$

and the gauge-invariant (under the spacetime integral) field strength is given by

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu - ig[A_\mu, A_\nu]_* \quad (48)$$

which generates the usual kinetic term $-\text{Tr}(\mathcal{F}^{\mu\nu} * \mathcal{F}_{\mu\nu})$.

This allows us to build the covariant derivative for Φ in the usual way for a field in the adjoint representation. Defining

$$D_\mu \Phi = \partial_\mu \Phi - ig[\mathcal{A}_\mu, \Phi]_* \quad (49)$$

the correctly normalized kinetic term for Φ now reads

$$\text{Tr}(D_\mu \Phi * D^\mu \Phi) = \text{Tr}(\partial_\mu \Phi \partial^\mu \Phi) - 2ig \text{Tr}(\partial_\mu \Phi * [\mathcal{A}^\mu, \Phi]_*) - g^2 \text{Tr}([\mathcal{A}^\mu, \Phi]_* * [\mathcal{A}_\mu, \Phi]_*) \quad (50)$$

using $\text{Tr}(A * [B, C]_*) = \text{Tr}([B, C]_* * A)$.

Due to the noncommutativity (and exacerbated by the fact that we have a non-Abelian gauge group), the complete expansion of some of these objects in component form yields a large number of terms, especially in the $\text{Tr}([\mathcal{A}, \Phi][\mathcal{A}, \Phi])$ term; because of our stated intention to study the mass renormalization of the Higgs at one-loop, we will only exhibit the terms we will later need.

Next, as long as we continue to consider translationally invariant vacua, the discussion of spontaneous symmetry breaking (at the classical level) proceeds exactly as in the previous section. Thus defining as before $\phi_3 = \sigma + a$, and $\pi = \phi$, we have successively

$$\text{Tr}(\partial_\mu \Phi \partial^\mu \Phi) = \frac{1}{2} [(\partial_\mu \sigma)^2 + (\partial_\mu \phi_4)^2] + \partial_\mu \pi \partial^\mu \pi^* \quad (51)$$

and

$$\begin{aligned} -2ig \text{Tr}(\partial_\mu \Phi [\mathcal{A}^\mu, \Phi]) &= -\frac{ig}{2} \left\{ \partial_\mu \sigma \left[[A_3^\mu, \phi_4] + [A_4^\mu, \sigma] + \{A^{*\mu}, \pi\} - \{\pi^*, A^\mu\} \right] \right. \\ &\quad + \partial_\mu \phi_4 \left[[A_3^\mu, \sigma] + [A_4^\mu, \phi_4] + [A^{*\mu}, \pi] + [A^\mu, \pi^*] \right] \\ &\quad + \partial_\mu \pi \left[[A_4^\mu, \pi^*] + [A^{*\mu}, \phi_4] + \{A_3^\mu, \pi^*\} - \{A^{*\mu}, \sigma\} \right] \\ &\quad + \partial_\mu \pi^* \left[[A_4^\mu, \pi] + [A^\mu, \phi_4] - \{A_3^\mu, \pi\} + \{A^\mu, \sigma\} \right] \\ &\quad \left. + 2a(\partial_\mu \pi^* A^\mu - \partial_\mu \pi A^{*\mu}) \right\} \end{aligned} \quad (52)$$

(the last two terms are the ones the R_ξ gauges are engineered to cancel) and

$$\begin{aligned} -g^2 \text{Tr}([\mathcal{A}^\mu, \Phi][\mathcal{A}_\mu, \Phi]) &= \frac{-g^2}{4} \left\{ A_3^\mu [\sigma, A_{3\mu}] \sigma + A_4^\mu [\sigma, A_{4\mu}] \sigma + A_3^\mu [\phi_4, A_{3\mu}] \phi_4 + A_4^\mu [\phi_4, A_{4\mu}] \phi_4 \right. \\ &\quad + A_3^\mu [\sigma, A_{4\mu}] \phi_4 + A_3^\mu [\phi_4, A_{4\mu}] \sigma + A_4^\mu [\phi_4, A_{3\mu}] \sigma + A_4^\mu [\sigma, A_{3\mu}] \phi_4 \\ &\quad + \left[2A^\mu \phi_4 A_\mu^* \phi_4 - \phi_4^2 (A_\mu^* A^\mu + A_\mu A^{*\mu}) \right] + 2(\phi_4 A^\mu \sigma A_\mu^* - \sigma A^\mu \phi_4 A_\mu^*) \\ &\quad \left. + \phi_4 \sigma [A^\mu, A_\mu^*] + \sigma \phi_4 [A^\mu, A_\mu^*] + 4a \phi_4 [A^\mu, A_\mu^*] \right\} \end{aligned}$$

$$\begin{aligned}
& + g^2 \left\{ \frac{1}{4} \left[2A^\mu \sigma A_\mu^* \sigma + \sigma^2 (A_\mu^* A^\mu + A_\mu A^{*\mu}) \right] \right. \\
& + a\sigma (A^\mu A_\mu^* + A_\mu^* A^\mu) + a^2 A_\mu^* A^\mu \left. \right\} + \dots
\end{aligned} \tag{53}$$

where we have written the terms which survive in the commutative limit in (53) at the end, and where the ellipsis represents four-field terms involving π and π^* that do not contribute to the one-loop corrections to the inverse Higgs/sigma propagator. The last term in (53) gives the complex gauge field A^μ its properly weighted mass $M \equiv ag$. The Feynman rules we will need from these interactions are displayed in the appendix.

The construction of the R_ξ gauge fixing proceeds as in the commutative case, because the gauge-fixing function is linear in the fields, so its Gaussian weighted insertion into the Lagrangian density is at most quadratic in the fields; by design, it is to cancel $A - \pi$ mixing terms. Explicitly, we take

$$\mathcal{G} \equiv \frac{1}{2\sqrt{\xi}} \begin{pmatrix} G_4 + G_3 & \sqrt{2}G^* \\ \sqrt{2}G & G_4 - G_3 \end{pmatrix} \tag{54}$$

with

$$\begin{aligned}
G &= \partial_\mu A^\mu - ig\xi a\pi & , & & G^* &= \partial_\mu A^{*\mu} + ig\xi a\pi^* \\
G_3 &= \partial_\mu A_3^\mu & , & & G_4 &= \partial_\mu A_4^\mu
\end{aligned} \tag{55}$$

so the contribution to the Lagrangian density is

$$\begin{aligned}
\mathcal{L}_{gf} = -\text{Tr}(\mathcal{G}^2) &= -\frac{1}{2\xi} \left[(\partial_\mu A_3^\mu)^2 + (\partial_\mu A_4^\mu)^2 \right] - \frac{1}{\xi} (\partial_\mu A^\mu)(\partial_\nu A^{*\nu}) \\
&+ iag(\partial_\mu \pi^* A^\mu - \partial_\mu \pi A^{*\mu}) - \xi a^2 g^2 \pi \pi^*
\end{aligned} \tag{56}$$

after integrating by parts the mixing terms and dropping the total derivative. This gives the usual gauge-dependent mass term to the would-be Goldstone mode, and signals that the π is now unphysical.

The final piece we need to complete the gauge-fixing of the Lagrangian is the ghost terms. Here however, we encounter a subtleties that must be dealt with carefully. We will not need to construct the entire ghost Lagrangian, since for our calculation, all we will need are the ghost propagators (corresponding to the gauge field A^μ), and their couplings to the physical Higgs, the latter of which are present even in the commutative Abelian Higgs model.

The first issue that arises is in the construction of the ghosts corresponding to the massive complex vector field in this theory. In commutative gauge theories, to each *real* gauge field (more precisely to each real gauge fixing function), corresponds a *complex* ghost. In the commutative case this poses no problem since we just decompose a complex gauge field into two real fields. But as we have seen in noncommutative gauge theories, this is

a more subtle issue. In particular, if we go to a real basis for A^μ , the off-diagonal massive complex vector field, then the gauge transformations δA^μ and $\delta A^{*\mu}$ will imply different gauge transformation laws for the two real components, which we putatively label A_1 and A_2 , due to the noncommutativity. The same argument applies to π , which occurs in the gauge fixing function G . Explicitly, if we read the (infinitesimal) gauge transformations for A_1 and A_2 from δA^μ we obtain

$$\begin{aligned}\delta A_1^\mu &= \frac{1}{4} [(A_4^\mu - A_3^\mu)\lambda_2 - \lambda_2(A_3^\mu + A_4^\mu) + A_2^\mu(\lambda_3 + \lambda_4) - (\lambda_4 - \lambda_3)A_2^\mu] + \frac{1}{2g}\partial^\mu\lambda_1 \\ \delta A_2^\mu &= \frac{1}{4} [\lambda_1(A_3^\mu + A_4^\mu) - (A_4^\mu - A_3^\mu)\lambda_1 + (\lambda_4 - \lambda_3)A_1^\mu - A_1^\mu(\lambda_3 + \lambda_4)] + \frac{1}{2g}\partial^\mu\lambda_2\end{aligned}\quad (57)$$

whereas if we read them from $\delta A^{*\mu}$, we obtain the same result but with the λ 's and A 's in the opposite order. Similar results are obtained for the decomposition of π . However, in the Fadeev-Popov construction of the ghosts, we are really interested in objects of the form $\delta G/\delta\lambda$, which, for the class of gauges we are studying, are *linear* in the fields, and so are independent of the star product⁵. So whether we read off the gauge transformations for A_1 and A_2 from δA^μ , or $\delta A^{*\mu}$ or even define the real component fields from $A^\mu + A^{*\mu}$ and $i(A^{*\mu} - A^\mu)$, the ‘derivatives’ of the gauge-fixing functions with respect to the infinitesimal gauge parameters are the same. Again, the same arguments hold for π . For our purposes, we need only note the presence of two complex ghosts denoted by c_1 and c_2 corresponding to the gauge field A^μ (or the gauge-fixing function G), and the ghost-ghost-Higgs couplings associated with these ghosts, which we now fix. We first decompose G

$$G = \partial_\mu A^\mu - ig\xi a\pi \rightarrow \begin{cases} G_1 = \partial_\mu A_1 + g\xi a\pi_2 \\ G_2 = \partial_\mu A_2 - g\xi a\pi_1 \end{cases}\quad (58)$$

Then from (41), and (57) we obtain

$$\begin{aligned}\frac{\delta\pi_1}{\delta\lambda_2} &= -\frac{1}{2}(\sigma + a) \\ \frac{\delta\pi_2}{\delta\lambda_1} &= \frac{1}{2}(\sigma + a) \\ \frac{\delta A_1^\mu}{\delta\lambda_1} &= \frac{\delta A_2^\mu}{\delta\lambda_2} = \frac{1}{2g}\partial^\mu\end{aligned}\quad (59)$$

Combining these, we get the derivatives

$$\frac{\delta G_1}{\delta\lambda_1} = \frac{\delta G_2}{\delta\lambda_2} = \frac{1}{2g}\partial_\mu\partial^\mu + \frac{1}{2}\xi ag(\sigma + a)\quad (60)$$

which yield the desired interactions (modulo a factor of $2g$, which we absorb into the definition of the ghosts themselves to obtain canonical kinetic terms), plus canonical ghost kinetic and mass terms:

$$\bar{c}_1 \left[-\partial_\mu\partial^\mu - \xi g^2 a(\sigma + a) \right] c_1 + (1 \leftrightarrow 2)\quad (61)$$

⁵We are abusing notation. The λ 's in all equations hitherto are implicitly infinitesimal, so are really $\delta\lambda$'s.

Of course there are other interactions between ghosts and gauge fields, because we have a (NC) non-Abelian group and arising from other possible derivatives, that we will not require.

The second issue that we fix by hand, is the orderings of the ghost-ghost-Higgs couplings. In commutative gauge theories, the ghost terms are written schematically as $\bar{c}(\delta G/\delta\lambda)c$, which includes terms of the form $\bar{c}\sigma c$. We have two possible noncommutative orderings for such terms, and as in the BRST treatment of unbroken noncommutative gauge theories (see for example [13]) we must include both orderings, symmetrically weighted. In the calculation we report below, we will find that in the absence of such symmetric weightings of the ghost orderings, the gauge-dependence of the renormalized theory which we will find, would be more severe. Thus the final piece of the Fadeev-Popov ghost contribution to the Lagrangian density that we will require is given by

$$\mathcal{L}_{FPG} = \bar{c}_1 \left[-\partial_\mu \partial^\mu - \xi M^2 \right] c_1 - \frac{\xi M^2}{2a} [\bar{c}_1 \sigma c_1 + \bar{c}_1 c_1 \sigma] + (1 \leftrightarrow 2) + \dots \quad (62)$$

where we use $M = ag$, and where the ellipsis denotes the aforementioned ghost-ghost-gauge couplings we do not need here.

Putting it all together, the classical R_ξ gauge-fixed Lagrangian density is given by

$$\mathcal{L} = -\text{Tr}(\mathcal{F}^{\mu\nu} \mathcal{F}_{\mu\nu}) + \text{Tr}(D_\mu \Phi D^\mu \Phi) + \mu^2 \text{Tr}(\Phi^2) - \lambda_1 \text{Tr}(\Phi^4) - \lambda_2 [\text{Tr}(\Phi^2)]^2 + \mathcal{L}_{gf} + \mathcal{L}_{FPG} \quad (63)$$

where star products are implicit. We now discuss the one-loop on-shell mass renormalization of the physical Higgs, which we have been denoting by σ . The discussion will be very similar to the commutative Abelian Higgs model. In fact, for this particular calculation, the differences between the commutative Abelian Higgs model and the *commutative* $U(2)$ adjoint representation Higgs model essentially amount to factors of two, arising from having a complex would-be Goldstone mode, a complex vector field, twice the number of ghost-ghost-Higgs couplings, etcetera. In particular, the non-Abelian nature of $U(2)$ plays no essential role at this order. The *noncommutative* $U(2)$ model we have defined is however completely different, as we will now show.

Following Appelquist et. al. [9], we generate the counterterms for this model by rescaling the fields and parameters according to

$$\begin{aligned} \mathcal{A}_\mu &\rightarrow \sqrt{Z_3} \mathcal{A}_\mu, \quad \Phi \rightarrow \sqrt{Z} \Phi, \quad g \rightarrow \frac{1}{\sqrt{Z_3}} g \\ \mu^2 &\rightarrow \frac{Z_\mu}{Z} \mu^2, \quad \lambda_1 \rightarrow \frac{Z_{\lambda_1}}{Z^2} \lambda_1, \quad \lambda_2 \rightarrow \frac{Z_{\lambda_2}}{Z^2} \lambda_2 \end{aligned} \quad (64)$$

applied to the symmetric Lagrangian (the gauge fixing terms, and the Fadeev-Popov terms are assumed to be written in terms of renormalized fields [9]). The only terms out of the counterterm Lagrangian we need are the terms associated with σ , and σ^2 :

$$\begin{aligned} &\left[Z_\mu \lambda - Z_{\lambda_1} \frac{\lambda_1}{2} - Z_{\lambda_2} \lambda_2 \right] a^3 \sigma + \frac{1}{2} \left[(Z_\mu - 1) \mu^2 - \frac{3}{2} (Z_{\lambda_1} - 1) \lambda_1 a^2 - 3 (Z_{\lambda_2} - 1) \lambda_2 a^2 \right. \\ &\left. + p^2 (Z - 1) \right] \sigma^2 \subset \mathcal{L}_{ct} \end{aligned} \quad (65)$$

At this point we note that Z_μ/Z must be a gauge-independent quantity, since it represents the (symmetric) mass renormalization of the model [9]. In the loop expansion,

$$\frac{Z_\mu}{Z} = \frac{1 + Z_\mu^{(1)} + O(\hbar^2)}{1 + Z^{(1)} + O(\hbar^2)} = 1 + (Z_\mu^{(1)} - Z^{(1)}) + O(\hbar^2) \quad (66)$$

which means that $Z_\mu - Z$ must be gauge-independent to lowest nontrivial order. But this quantity, as we now show, is proportional to the on-shell mass renormalization of the physical Higgs σ . Without imposing a condition on the one-point Higgs amplitude (which is itself gauge-dependent, and divergent but unphysical), there are two types of one-loop quantum corrections to the inverse Higgs propagator: the usual 1PI self-energy graphs (and their counterterm), and one-point Higgs tadpole insertions (and their counterterm). From (65) these two counterterms are

$$\begin{aligned} \text{---}\sigma\text{---}\bigtimes\text{---}\sigma &= i \left[(Z_\mu - 1)\mu^2 - \frac{3}{2}(Z_{\lambda_1} - 1)\lambda_1 a^2 - 3(Z_{\lambda_2} - 1)\lambda_2 a^2 + p^2(Z - 1) \right] \\ \text{---}\sigma\text{---}\bigvee\text{---}\sigma &= -6i\lambda a \frac{i}{-2\lambda a^2} i \left[Z_\mu \lambda - \frac{\lambda_1}{2} Z_{\lambda_1} - \lambda_2 Z_{\lambda_2} \right] a^3 \\ &= -3ia^2 \left[Z_\mu \lambda - \frac{\lambda_1}{2} Z_{\lambda_1} - \lambda_2 Z_{\lambda_2} \right] \end{aligned} \quad (67)$$

where we have used the fact that no external momentum flows into the internal σ propagator to collapse the noncommutative $3 - \sigma$ vertex to $-6i\lambda a$ in the latter counterterm. Using $\lambda \equiv \lambda_1/2 + \lambda_2$, the sum of these two counterterms equals

$$\sum_{ct} = -i \left[2\lambda a^2 (Z_\mu - 1) - p^2 (Z - 1) \right] \quad (68)$$

which, when evaluated on the mass-shell of the Higgs, $p^2 = 2\mu^2 = 2\lambda a^2$, equals

$$\sum_{ct}(p^2 = 2\mu^2) = -2\lambda a^2 i [Z_\mu - Z] \quad (69)$$

Thus, denoting $\Pi(p^2)$ as the sum of all 1PI and one-point tadpole corrections to the inverse σ propagator, performing on-shell mass subtraction means that

$$\Pi(2\mu^2) = 2\lambda a^2 i [Z_\mu - Z] \quad (70)$$

and so should be gauge-invariant by the previous argument.

We now come to the main calculation of this paper, wherein we show that $\Pi(2\mu^2)$ is *not* gauge-invariant at the one-loop level for the noncommutative theory. More specifically we will show that the divergent (i.e. cutoff dependent⁶) part of this graphical sum is not

⁶Of course, we will be using dimensional regularization which respects gauge symmetries.

gauge-invariant even when evaluated on-shell. Because the nonplanar parts of these one-loop graphs are finite, in essence we will only be examining the now 're-weighted' planar parts of these graphs. Furthermore, because we will not evaluate, nor keep track of these nonplanar pieces (which become divergent themselves as $\theta \rightarrow 0$) for simplicity, we will not be able to take a manifest commutative limit at the end. Nonetheless, we wish to compare with the commutative limit (if only as a double check that we have the correct graphs), so to this end, after each graph we will write the commutative counterpart with little effort. Finally, we will proceed as far as possible algebraically (by getting momentum independent pieces separately from $O(p^2)$ pieces) in order to see how most of the ξ (i.e. gauge parameter) dependence is still cancelled. The presence of divergent wave-function renormalizations due to momentum-dependent vertices (i.e. outside of the noncommutative phases) will not allow us to completely and conveniently carry this out, so we will express all remaining divergences in terms of the dimensional pole at $D = 4$.

We will handle the purely noncommutative graphs at the end; first we will calculate the 1PI graphs that survive in the commutative limit. We use the definition

$$I_n(m^2) = \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 - m^2)^n} \quad (71)$$

which will be useful for the momentum-independent pieces, the identity $\Gamma(2 - D/2) = (1 - D/2)\Gamma(1 - D/2) = -\Gamma(1 - D/2) + \text{finite}$, near $D = 4$, and the dimensional regularization formulae

$$\begin{aligned} \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 - \Delta)^n} &= \frac{(-1)^n i \Gamma(n - D/2)}{(4\pi)^{D/2} \Gamma(n)} \Delta^{D/2-n} \\ \int \frac{d^D k}{(2\pi)^D} \frac{k^2}{(k^2 - \Delta)^n} &= \frac{(-1)^{n-1} i D \Gamma(n - D/2 - 1)}{(4\pi)^{D/2} 2 \Gamma(n)} \Delta^{D/2-n+1} \\ \int \frac{d^D k}{(2\pi)^D} \frac{(k^2)^2}{(k^2 - \Delta)^n} &= \frac{(-1)^n i D(D+2) \Gamma(n - D/2 - 2)}{(4\pi)^{D/2} 4 \Gamma(n)} \Delta^{D/2-n+2} \end{aligned} \quad (72)$$

Thus

$$\begin{aligned} \text{Diagram: } \sigma \xrightarrow{p} \text{---} \bullet \text{---} \xrightarrow{p} \sigma \text{ with a loop } k \text{ on the vertex} &= -\frac{2i\lambda}{2} \int \frac{d^D k}{(2\pi)^D} \left(2 \cos^2\left(\frac{p \times k}{2}\right) + 1 \right) \frac{i}{k^2 - 2\lambda a^2} \\ &= \lambda \int \frac{d^D k}{(2\pi)^D} \left(2 + \cos\left(\frac{p \times k}{2}\right) \right) \frac{1}{k^2 - 2\lambda a^2} \\ &= 2\lambda I_1(2\lambda a^2) + \text{finite} \\ &= \frac{2i\lambda(2\lambda a^2)}{(4\pi)^2} \Gamma(2 - D/2) + \text{finite} \end{aligned} \quad (73)$$

This gauge independent, and in the commutative theory is equal to $3\lambda I_1(2\lambda a^2)$.

$$\begin{aligned}
\text{Diagram: } \sigma \xrightarrow{p} \bullet \xrightarrow{k} \bullet \xrightarrow{p} \sigma &= \frac{1}{2} \int \frac{d^D k}{(2\pi)^D} \frac{[-2i(\lambda_1 + \lambda_2) - i\lambda_1 \cos(p \times k)] i}{k^2 - \lambda_1 a^2} \\
&= (\lambda_1 + \lambda_2) I_1(\lambda_1 a^2) \\
&= (\lambda_1 + \lambda_2) \frac{i(\lambda_1 a^2)}{(4\pi)^2} \Gamma(2 - D/2) + \text{finite}
\end{aligned} \tag{74}$$

This is gauge independent, and in the commutative theory is equal to $[(3/2)\lambda_1 + \lambda_2] I_1(\lambda_1 a^2)$.

$$\begin{aligned}
\text{Diagram: } \sigma \xrightarrow{p} \bullet \xrightarrow{k} \bullet \xrightarrow{k+p} \sigma &= \frac{(-2i\lambda a)^2}{2} \int \frac{d^D k}{(2\pi)^D} \frac{\left[3 \cos\left(\frac{p \times k}{2}\right)\right]^2 i^2}{(k^2 - 2\lambda a^2)[(p+k)^2 - 2\lambda a^2]} \\
&= 9\lambda^2 a^2 \int \frac{d^D k}{(2\pi)^D} \frac{1 + \cos(p \times k)}{(k^2 - 2\lambda a^2)^2} + \text{finite} \\
&= 9\lambda \mu^2 I_2(2\lambda a^2) + \text{finite} \\
&= \frac{9i\lambda \mu^2}{(4\pi)^2} \Gamma(2 - D/2) + \text{finite}
\end{aligned} \tag{75}$$

This is gauge independent, and in the commutative theory is equal to $18\lambda \mu^2 I_2(2\lambda a^2) + \text{finite}$.

$$\begin{aligned}
\text{Diagram: } \sigma \xrightarrow{p} \bullet \xrightarrow{k} \bullet \xrightarrow{k+p} \sigma &= \frac{[-2i(\lambda + \lambda_1)a]^2}{2} \int \frac{d^D k}{(2\pi)^D} \frac{\cos^2\left(\frac{p \times k}{2}\right) i^2}{(k^2 - \lambda_1 a^2)[(p+k)^2 - \lambda_1 a^2]} \\
&= (\lambda + \lambda_1)^2 a^2 \int \frac{d^D k}{(2\pi)^D} \frac{1 + \cos(p \times k)}{(k^2 - \lambda_1 a^2)^2} + \text{finite} \\
&= (\lambda + \lambda_1)^2 a^2 I_2(\lambda_1 a^2) + \text{finite} \\
&= \frac{i(\lambda + \lambda_1)^2 a^2}{(4\pi)^2} \Gamma(2 - D/2) + \text{finite}
\end{aligned} \tag{76}$$

This is gauge independent, and in the commutative theory is equal to $2(\lambda + \lambda_1)^2 a^2 I_2(\lambda_1 a^2) + \text{finite}$.

$$\text{Diagram: } \sigma \xrightarrow{p} \bullet \xrightarrow{k} \bullet \xrightarrow{k+p} \sigma = (-2i\lambda a)^2 \int \frac{d^D k}{(2\pi)^D} \frac{\cos^2\left(\frac{p \times k}{2}\right) i^2}{(k^2 - \xi M^2)[(p+k)^2 - \xi M^2]}$$

$$\begin{aligned}
&= 2\lambda\mu^2 I_2(\xi M^2) + \text{finite} \\
&= \frac{2i\lambda\mu^2}{(4\pi)^2} \Gamma(2 - D/2) + \text{finite}
\end{aligned} \tag{77}$$

There is no symmetry factor because π is complex. The divergent part of this graph is gauge-independent, and in the commutative theory this graph is equal to $4\lambda\mu^2 I_2(\xi M^2) + \text{finite}$.

The first graph we encounter with a gauge-dependent divergent piece is

$$\begin{aligned}
\text{Diagram: } \sigma \xrightarrow{p} \text{---} \text{---} \text{---} \text{---} \xrightarrow{p} \sigma \quad \text{with a dashed loop of momentum } k \text{ on the internal line} &= \int \frac{d^D k}{(2\pi)^D} [-2i(\lambda_1 + \lambda_2) + i\lambda_1 \cos(p \times k)] \frac{i}{k^2 - \xi M^2} \\
&= 2(\lambda_1 + \lambda_2) I_1(\xi M^2) + \text{finite} \\
&= \frac{2i(\lambda_1 + \lambda_2)(\xi M^2)}{(4\pi)^2} \Gamma(2 - D/2) + \text{finite}
\end{aligned} \tag{78}$$

In the commutative theory, this graph is given by

$$2\lambda I_1(\xi M^2) = \frac{2i(\frac{\lambda_1}{2} + \lambda_2)\xi M^2}{(4\pi)^2} \Gamma(2 - D/2) + \text{finite} \tag{79}$$

Next we have the 1PI graphs with the complex vector field propagating in the loop. They all have gauge-dependent divergent parts.

$$\begin{aligned}
\text{Diagram: } \sigma \xrightarrow{p} \text{---} \text{---} \text{---} \text{---} \xrightarrow{p} \sigma \quad \text{with a wavy loop of momentum } k \text{ and } k+p &= (2ia g^2)^2 (-i)^2 \int \frac{d^D k}{(2\pi)^D} \cos^2\left(\frac{p \times k}{2}\right) \left\{ \frac{D - 2k^2/M^2 + k^4/M^4}{(k^2 - M^2)^2} \right. \\
&\quad \left. + \frac{2(k^2/M^2 - k^4/M^4)}{(k^2 - M^2)(k^2 - \xi M^2)} + \frac{k^4/M^4}{(k^2 - \xi M^2)^2} \right\} + \text{finite} \\
&= 2g^4 a^2 [(D - 1)I_2(M^2) + \xi^2 I_2(\xi M^2)] + \text{finite}
\end{aligned} \tag{80}$$

In the commutative theory this graph is given by

$$4g^4 a^2 [(D - 1)I_2(M^2) + \xi^2 I_2(\xi M^2)] + \text{finite} \tag{81}$$

$$\begin{aligned}
\text{Diagram: } \sigma \xrightarrow{p} \text{---} \text{---} \text{---} \text{---} \xrightarrow{p} \sigma \quad \text{with a wavy loop of momentum } k &= ig^2 (-i) \int \frac{d^D k}{(2\pi)^D} [1 + \cos(p \times k)] \left\{ \frac{D - k^2/M^2}{k^2 - M^2} + \frac{k^2/M^2}{k^2 - \xi M^2} \right\} \\
&= g^2 [(D - 1)I_1(M^2) + \xi I_1(\xi M^2)] + \text{finite}
\end{aligned} \tag{82}$$

In the commutative theory this graph is given exactly by

$$2g^2 [(D-1)I_1(M^2) + \xi I_1(\xi M^2)] \quad (83)$$

The next (type of) graph is the most difficult we will encounter, due to the presence of momentum-dependent (outside of the noncommutative phase) σ - π - A^* and σ - π^* - A vertices, which yield divergent contributions at $O(p^2)$; i.e. divergent wavefunction renormalization, as can be seen by power-counting. We need the following identity which is most easily established by introducing a Feynman parameter, and symmetrically integrating:

$$\begin{aligned} & \frac{d}{dp^2} \left(\int \frac{d^D k}{(2\pi)^D} \left\{ \frac{(2p+k)^2 - (2p \cdot k + k^2)^2/M^2}{[(p+k)^2 - \xi M^2](k^2 - M^2)} + \frac{(2p \cdot k + k^2)^2/M^2}{[(p+k)^2 - \xi M^2](k^2 - \xi M^2)} \right\} \right) \\ &= \frac{i}{(4\pi)^2} \Gamma(2 - D/2)(3 - \xi) + \text{finite} \end{aligned} \quad (84)$$

We then Taylor expand the planar part of the graph in powers of p^2 (as we have done implicitly in all of the previous graphs where the external momentum circulated in the loop, but which only contributed to the finite parts which we explicitly neglect). Thus

$$\begin{aligned} \text{Diagram: } \sigma \xrightarrow{p} \pi \text{---} \text{Loop} \text{---} A^\nu \xrightarrow{\sigma} &= (ig)^2 i(-i) \int \frac{d^D k}{(2\pi)^D} \cos^2\left(\frac{p \times k}{2}\right) \frac{(-2p-k)_\mu (-2p-k)_\nu}{(p+k)^2 - \xi M^2} \times \\ & \quad \left\{ \frac{g^{\mu\nu} - k^\mu k^\nu/M^2}{k^2 - M^2} + \frac{k^\mu k^\nu}{k^2 - \xi M^2} \right\} \\ &= -\frac{g^2}{2} \int \frac{d^D k}{(2\pi)^D} \left\{ \frac{k^2 - k^4/M^2}{(k^2 - M^2)(k^2 - \xi M^2)} + \frac{k^4/M^2}{(k^2 - \xi M^2)^2} \right\} \\ & \quad - \frac{1}{2} \frac{ig^2 p^2}{(4\pi)^2} (3 - \xi) + \text{finite} \\ &= -\frac{g^2}{2} [\xi I_1(\xi M^2) + \xi^2 M^2 I_2(\xi M^2)] - \frac{1}{2} \frac{ig^2 p^2}{(4\pi)^2} (3 - \xi) \Gamma(2 - D/2) + \text{finite} \end{aligned} \quad (85)$$

As in the previous two graphs, the divergent piece of this graph in the commutative theory is twice this:

$$-g^2 [\xi I_1(\xi M^2) + \xi^2 M^2 I_2(\xi M^2)] - \frac{ig^2 p^2}{(4\pi)^2} (3 - \xi) \Gamma(2 - D/2) + \text{finite} \quad (86)$$

There is a second graph (a ‘crossing’) with the σ - π - A^* and σ - π^* - A vertices switched with respect to the external lines; equivalently, the gauge charge circulates in the opposite direction. It has the identical value as the graph just considered, so we will simply multiply this result by two in our later accounting.

Next are the ghost graphs, which, remembering to include an overall minus sign, are given by:

$$\begin{aligned}
\text{Diagram: } \sigma \xrightarrow[p]{c_{1(2)}} \text{ghost loop} \xrightarrow[k+p]{\sigma} &= 2(-1) \left(-i \frac{\xi M^2}{a} \right)^2 i^2 \int \frac{d^D k}{(2\pi)^D} \frac{\cos^2(\frac{p \times k}{2})}{[(p+k)^2 - \xi M^2](k^2 - M^2)} \\
&= -g^4 a^2 \xi^2 I_2(\xi M^2) + \text{finite}
\end{aligned} \tag{87}$$

the factor of two in the first line coming from the fact that we have two sets of ghosts. The graphs in the commutative theory are

$$-2g^4 a^2 \xi^2 I_2(\xi M^2) + \text{finite} \tag{88}$$

Note that if we had not included both ghost-ghost-Higgs orderings, the noncommutative phase at each vertex would cancel, and the ghost graphs would coincide with their commutative counterparts. Clearly, the ghost graph is used to cancel divergent, momentum-independent, gauge-dependent contributions coming from (80) and (85), whose divergent pieces in the noncommutative case are half those of their commutative counterparts. Thus, if we were not to introduce both orderings (and symmetrically weight them), we would (already) obtain a manifest gauge-dependence proportional to $\xi^2 g^2 M^2$.

We have four purely noncommutative 1PI graphs which we will add later. Let us now consider the one-point tadpole contributions. They are manifestly the same for both the noncommutative and commutative theories, because no external momentum flows into the tadpole, so all vertex phases degenerate to one.

There are two gauge-independent tadpoles:

$$\begin{aligned}
\text{Diagram: } \sigma \xrightarrow[p]{\text{tadpole}} \sigma &= (-6i\lambda a) \frac{i}{-2\lambda a^2} (-6i\lambda a) \frac{i}{2} I_1(2\mu^2) \\
&= -9\lambda I_1(2\mu^2)
\end{aligned} \tag{89}$$

and

$$\begin{aligned}
\text{Diagram: } \sigma \xrightarrow[p]{\text{ghost tadpole}} \sigma &= (-6i\lambda a) \frac{i}{-2\lambda a^2} [-2i(\lambda + \lambda_1)a] \frac{i}{2} I_1(\lambda_1 a^2) \\
&= -3(\lambda + \lambda_1) I_1(\lambda_1 a^2)
\end{aligned} \tag{90}$$

The gauge-dependent one-point tadpole graphs are

$$\begin{aligned}
\text{Diagram: } \sigma \text{ --- } p \text{ --- } \text{tadpole} \text{ --- } p \text{ --- } \sigma &= (-6i\lambda a) \frac{i}{-2\lambda a^2} (-2i\lambda a) i I_1(\xi M^2) \\
&= -6\lambda I_1(\xi M^2) \\
&= \frac{-6i\lambda}{(4\pi)^2} (\xi g^2 a^2) \Gamma(2 - D/2) + \text{finite} \tag{91}
\end{aligned}$$

and

$$\begin{aligned}
\text{Diagram: } \sigma \text{ --- } p \text{ --- } \text{loop} \text{ --- } p \text{ --- } \sigma &= (-6i\lambda a) \frac{i(-i)}{-2\lambda a^2} 2ia g^2 \int \frac{d^D k}{(2\pi)^D} \frac{D - k^2/M^2}{k^2 - M^2} + \frac{k^2/M^2}{k^2 - \xi M^2} \\
&= -6g^2 \left[(D-1) I_1(M^2) + \xi I_1(\xi M^2) \right] \tag{92}
\end{aligned}$$

and

$$\begin{aligned}
\text{Diagram: } \sigma \text{ --- } p \text{ --- } \text{ghost loop} \text{ --- } p \text{ --- } \sigma &= 2 \times (-1) (-6i\lambda a) \frac{i}{-2\lambda a^2} (-i\xi g^2 a) i I_1(\xi M^2) \\
&= +6\xi g^2 I_1(\xi M^2) \tag{93}
\end{aligned}$$

Note that the gauge-dependence between the last two graphs explicitly cancels. Thus the only one-point tadpole correction which will now contribute to our gauge-dependence calculation is (91).

This completes the list of graphs that survive in the commutative limit. Thus let us first check that the sum of divergent gauge-dependent contributions for the commutative theory graphs vanish on the Higgs mass-shell as required. Adding (79), (81), (83), $2 \times$ (86), (88), and (91)-(93) we obtain

$$\begin{aligned}
\Pi_{\xi\text{-dep, div, comm}}(p^2) &= 2\Gamma\lambda\xi M^2 + 4g^4 a^2 \xi^2 I_2(\xi M^2) + 2g^2 \xi I_1(\xi M^2) - 2g^2 \left[\xi I_1(\xi M^2) \right. \\
&\quad \left. + \xi^2 M^2 I_2(\xi M^2) \right] + 2\Gamma\xi g^2 p^2 - 2g^4 a^2 \xi^2 I_2(\xi M^2) - 6\Gamma\lambda g^2 a^2 \\
&= 2\xi\Gamma g^2 (p^2 - 2\lambda a^2) \\
&\rightarrow 0 \quad \text{as } p^2 \rightarrow 2\lambda a^2 \tag{94}
\end{aligned}$$

where Γ is shorthand for $i\Gamma(2 - D/2)/(16\pi^2)$.

Repeating this calculation for the noncommutative theory, by adding the gauge-dependent, divergent pieces from (78), (80), (82), $2 \times$ (85), (87), (91)-(93) we get

$$\sum_{\xi\text{-dep, div, noncomm}} = 2(\lambda_1 + \lambda_2)\Gamma\xi M^2 + 2g^4 a^2 \xi^2 I_2(\xi M^2) + g^2 \xi I_1(\xi M^2) - g^2 \left[\xi I_1(\xi M^2) + \right.$$

$$\begin{aligned}
& + \xi^2 M^2 I_2(\xi M^2) \Big] + \Gamma \xi g^2 p^2 - g^4 a^2 \xi^2 I_2(\xi M^2) - 6 \Gamma \lambda g^2 a^2 \\
& = \xi \Gamma g^2 (p^2 - \lambda_1 a^2 - 4 \lambda_2 a^2) \\
& \rightarrow -2 \xi \Gamma g^2 \lambda_2 a^2 \quad \text{as } p^2 \rightarrow 2 \lambda a^2
\end{aligned} \tag{95}$$

Thus, although most of the ξ cancellation persists, since the divergent parts of several of the graphs are simply halved with respect to their commutative counterparts, (78) is split differently than the p^2 wavefunction renormalization piece in (85), and (91), the one-point tadpole, is not split at all.

However, we still have to add the contributions from purely noncommutative graphs, i.e. graphs that disappear in the commutative limit. There are four of them, although only one will contribute. First the graphs with the gauge fields A_3^μ and A_4^μ as 1PI tadpoles disappear in dimensional regularization because they involve massless propagators:

$$\begin{array}{c} \sigma \\ \rightarrow \\ p \end{array} \xrightarrow{A_{3(4)}^\mu} \begin{array}{c} \text{loop} \\ k \end{array} \xrightarrow{p} \begin{array}{c} \sigma \\ \rightarrow \end{array} = 0 \tag{96}$$

The last two graphs we need to consider both involve the following integral (again evaluated by introducing a Feynman parameter, and symmetrically integrating)

$$\begin{aligned}
& \int \frac{d^D k}{(2\pi)^D} \frac{(2p+k)^2 + (\xi-1)(2p \cdot k + k^2)^2/k^2}{[(p+k)^2 - m^2]k^2} \\
& = \frac{i}{(4\pi)^2} \Gamma(2-D/2) [\xi m^2 + (3-\xi)p^2] + \text{finite}
\end{aligned} \tag{97}$$

which like (85) have divergent wavefunction renormalization contributions, and originate from the matter covariant derivative. Thus

$$\begin{aligned}
\begin{array}{c} \sigma \\ \rightarrow \\ p \end{array} \xrightarrow{A_3} \begin{array}{c} \text{loop} \\ k \end{array} \xrightarrow{k+p} \begin{array}{c} \sigma \\ \rightarrow \end{array} & = g^2 \int \frac{d^D k}{(2\pi)^D} \sin^2\left(\frac{p \times k}{2}\right) \frac{(-p-k-p)_\mu (p+k+p)_\nu i(-i)}{[(p+k)^2 - \lambda_1 a^2]k^2} \times \\
& \times \left[g^{\mu\nu} + (\xi-1) \frac{k^\mu k^\nu}{k^2} \right] \\
& = -\frac{g^2}{2} \int \frac{d^D k}{(2\pi)^D} [1 - \cos(p \times k)] \frac{(2p+k)^2 + (\xi-1)(2p \cdot k + k^2)^2/k^2}{[(p+k)^2 - \lambda_1 a^2]k^2} \\
& = -\frac{ig^2}{2(4\pi)^2} \Gamma(2-D/2) [\xi \lambda_1 a^2 + (3-\xi)p^2] + \text{finite}
\end{aligned} \tag{98}$$

and similarly

$$\begin{array}{c} \sigma \\ \rightarrow \\ p \end{array} \xrightarrow{A_4} \begin{array}{c} \text{loop} \\ k \end{array} \xrightarrow{k+p} \begin{array}{c} \sigma \\ \rightarrow \end{array} = g^2 \int \frac{d^D k}{(2\pi)^D} \sin^2\left(\frac{p \times k}{2}\right) \frac{(-p-k-p)_\mu (p+k+p)_\nu i(-i)}{[(p+k)^2 - 2\lambda a^2]k^2} \times$$

$$\begin{aligned}
& \times \left[g^{\mu\nu} + (\xi - 1) \frac{k^\mu k^\nu}{k^2} \right] \\
& = -\frac{ig^2}{2(4\pi)^2} \Gamma(2 - D/2) \left[\xi(2\lambda a^2) + (3 - \xi)p^2 \right]
\end{aligned} \tag{99}$$

In the notation used above, the sum of the divergent gauge-dependent pieces from these two graphs is

$$\begin{aligned}
\sum_{\xi\text{-dep, div}} (\text{pure noncomm}) &= \frac{\xi \Gamma g^2}{2} \left[(p^2 - \lambda_1 a^2) + (p^2 - 2\lambda a^2) \right] \\
&\rightarrow \xi \Gamma g^2 \lambda_2 a^2 \quad \text{as } p^2 \rightarrow 2\lambda a^2
\end{aligned} \tag{100}$$

which is not enough to cancel the residual piece in (95), although yet again, depends only on the coupling λ_2 (and g) and not λ_1 . Thus the sum of all gauge-dependent pieces evaluated on the Higgs mass-shell is

$$\Pi_{\xi\text{-dep, noncomm}}(p^2 = 2\lambda a^2) = -\xi \Gamma g^2 \lambda_2 a^2 + \text{finite} \tag{101}$$

where $\Gamma = i\Gamma(2 - D/2)/(16\pi^2)$.

This signals gauge-dependence in the on-shell mass renormalization of the Higgs in this model.

5 Discussion

This final result of the last section is our main result, and it is noteworthy for several reasons. First, like our discussion of the global theory, it depends only on the coupling to $\text{Tr}(\Phi^2)^2$ (as well as the gauge coupling), and not on the coupling to $\text{Tr}(\Phi^4)$. Furthermore this noncancellation of ξ dependence is more robust than it might appear. If we modify any of the noncommutative graphs that do have nonzero commutative limits, the result is generally made worse because a λ_1 dependence will be reintroduced. (This is why we kept the commutative calculation in parallel.) The purely noncommutative graph that does contribute, (98) has half the weight of (85) + crossing because it involves the *real* gauge boson A_3 , and the real field ϕ_4 , so we have not made an obvious ‘factor of two error’ that would kill the remaining ξ dependence; in particular, the presence of new purely noncommutative interactions does not save us, though no dependence on λ_1 is introduced by them. Also, as noted after (87), had we not included both orderings of the ghost-ghost-Higgs couplings, the result would be made worse by a residual $\xi^2 g^2 M^2$ dependence. As we would expect, the result is also proportional to a^2 , which of course is the order-parameter for spontaneous symmetry breaking.

Let us now generalize these results. First, as mentioned when we introduced the scalar potential for the NC $U(2)$ adjoint matter model, we did not include trace invariants involving

an odd number of Φ s so as to simplify the discussion of classical spontaneous symmetry breaking. In grand unified theories involving $SU(N)$ groups, an additional discrete symmetry under $\Phi \rightarrow -\Phi$ is usually imposed to exclude (in combination with the tracelessness of the $SU(N)$ adjoint) these terms and arrive at the form of the potential we consider. Because we have a $U(N)$ group however, so that $\text{Tr}(\Phi) = \phi_4 \neq 0$, the imposition of this discrete symmetry still permits the terms

$$V_2(\Phi) = \mu_2^2 \text{Tr}(\Phi) * \text{Tr}(\Phi) + \lambda_3 \text{Tr}(\Phi) * \text{Tr}(\Phi^3). \quad (102)$$

(we only include interactions renormalizable in the commutative theory), which come with coupling constants of mass dimension two and zero respectively. We give two arguments why these terms are irrelevant for our calculation. The general argument is simply that because each invariant comes with its own coupling constant, the λ_2 noncancellation of the previous section cannot be modified, in the same way that a residual ξ dependence also proportional to λ_1 (had we found one), would not affect the λ_2 noncancellation (in either the global or gauge theory). Put another way, the one-loop corrections to the inverse σ propagator are linear and additive in the scalar couplings, and the ξ dependence from each must disappear independently. (This argument applies irrespective of whether or not we have imposed the discrete symmetry.) In fact, the general inclusion of such terms in the scalar potential can only exacerbate the result by possibly introducing new independent ξ residuals. The second argument lies in the manifest expansion of the terms in (102), assuming now that we have imposed $\Phi \rightarrow -\Phi$ invariance. The first contributes an irrelevant mass term shift to ϕ_4 , and cannot affect the $\xi\lambda_2$ noncancellation, while the expansion of the second term, before shifting ϕ_3 , yields

$$\text{Tr}(\Phi) * \text{Tr}(\Phi^3) = \frac{1}{4}\phi_3^2\phi_4^2 + \frac{1}{8}\phi_3\phi_4\phi_3\phi_4 + (\pi\pi^*\phi_3\phi_4 \text{ and } \pi\pi^*\phi_4\phi_4 \text{ terms}) \quad (103)$$

The ‘ $\phi_3^2\phi_4^2$ ’ type-terms yield gauge-independent mass renormalizations of ϕ_3 at one loop exactly as in (74), [and (76)], while none of the other four-field terms contribute to the mass renormalization of ϕ_3 (and σ) at *one-loop* because they involve ϕ_4 . Finally, we note that if we do include the terms in (102), the discussion of classical spontaneous symmetry breaking by a translationally invariant vacuum, is effectively unmodified since

$$V_2(a, b) = \frac{\mu_2^2}{4}(a+b)^2 + \frac{\lambda_3}{16}(a+b)(a^3+b^3) \quad (104)$$

simply discards the vacua $a = b$ [discussed below (25), and which did not reflect spontaneously broken states], because they obviously no longer represent minima, thereby leaving the vacua $a = -b$, that we actually considered. This is a consequence of the discrete symmetry which now forbids constant shifts in the ϕ_4 field. Thus the only consequence of (102) for our purposes, is a harmless shift in the definition of the VEV a . From these arguments we conclude that our results actually hold for arbitrary (commutative-renormalizable) scalar potentials.

Next, consider the extension to the general $U(N)$ case. As we noted at the beginning of the previous section, the particular calculation exhibited there parallels that of the (commutative) Abelian Higgs model in form, because the now non-Abelian group essentially

contributes factors of two at one-loop. Higher rank groups admit more complicated adjoint representation symmetry breaking patterns (another reason we studied NC $U(2)$), but nonetheless the essence of our calculation should persist there by the same argument.

Thus we conclude that the mass-shell renormalization of the σ is gauge-dependent at the one-loop quantum level. Put another way, the quantum theory does not respect the gauge invariance of the classical theory, suggesting some sort of noncommutative anomaly, and that despite our best efforts (i.e. by going to the adjoint representation of a noncommutative ‘safe’ [8] gauge group), the σ is not a physical degree of freedom.

Let us reflect on the more general nature of these results. In both the (putative) spontaneously broken global theory and the gauge theory, certain ordinary quantum field theory cancellations, required for (continuum) renormalizability in the former case, and gauge-independence in the latter, no longer hold essentially as a consequence of the failure of one-point functions to see the noncommutativity (at least at one-loop). We remind the reader, that one-point amplitudes set the corrections to the order parameter for spontaneous symmetry breaking. In the global theory this is most easily manifested by the failure of the Goldstone’s theorem to hold at the quantum level, and in the gauge-theory by the appearance of a gauge-dependent (divergent) mass renormalization to what should be a physical degree of freedom. While going to the adjoint representation has partially alleviated the problem in the global case at the quantum level by introducing a new noncommutative interaction, and allows us to define a consistent *classical* gauge theory with spontaneous symmetry breaking, the remaining dependence in both models on the coupling to $\text{Tr}(\Phi^2)^2$ is both intriguing, and fatal. Again, if we turn off this interaction in the scalar potential by tuning λ_2 to zero at tree-level, we radiatively generate it, so we expect that these problems will resurface at two-loops, though an explicit calculation to verify this would be laborious, since the rest of the theory would have to be fixed at the one-loop level first.

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A Appendix

A.1 Propagators

$$\frac{\sigma}{\xrightarrow[p]{\sigma}} = \frac{i}{p^2 - 2\lambda a^2 + i\epsilon} \quad (105)$$

$$\frac{\pi}{\xrightarrow[p]{\pi}} = \frac{i}{p^2 - \xi M^2 + i\epsilon} \quad (106)$$

$$\frac{\phi_4}{\xrightarrow[p]{\phi_4}} = \frac{i}{p^2 - \lambda_1 a^2 + i\epsilon} \quad (107)$$

$$\frac{A^\mu}{\xrightarrow[p]{A^\mu}} = -i \left[\frac{g_{\mu\nu} - k_\mu k_\nu / M^2}{k^2 - M^2 + i\epsilon} + \frac{k_\mu k_\nu / M^2}{k^2 - \xi M^2 + i\epsilon} \right] \quad (108)$$

$$\frac{A_{3(4)}^\mu}{\xrightarrow[p]{A_{3(4)}^\mu}} = \frac{-i}{k^2 + i\epsilon} \left[g_{\mu\nu} + (\xi - 1) \frac{k_\mu k_\nu}{k^2} \right] \quad (109)$$

$$\frac{c_{1(2)}}{\xrightarrow[p]{c_{1(2)}}} = \frac{i}{p^2 - \xi M^2 + i\epsilon} \quad (110)$$

$$(111)$$

A.2 Scalar Potential Feynman Rules

All momenta flow into the vertices.

$$\begin{array}{c} \pi^* p_1 \quad \pi p_2 \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ \pi^* p_3 \quad \pi p_4 \end{array} = -2i(\lambda_1 + \lambda_2) \cos\left(\frac{p_1 \times p_2}{2} + \frac{p_3 \times p_4}{2}\right) - 2i\lambda_2 \cos\left(\frac{p_1 \times p_3}{2}\right) \cos\left(\frac{p_2 \times p_4}{2}\right) \quad (112)$$

$$\begin{array}{c} \pi^* p_1 \\ \diagdown \\ \bullet \\ \diagup \\ \pi p_2 \\ \diagup \\ \sigma p_3 \\ \diagdown \\ \sigma p_4 \end{array} = -2i(\lambda_1 + \lambda_2) \cos\left(\frac{p_1 \times p_2}{2}\right) \cos\left(\frac{p_3 \times p_4}{2}\right) + i\lambda_1 \cos\left(\frac{p_1 \times p_3}{2} + \frac{p_2 \times p_4}{2}\right) \quad (113)$$

$$\begin{array}{c} \pi^* p_1 \\ \diagdown \\ \bullet \\ \diagup \\ \pi p_2 \\ \diagup \\ \phi_4 p_3 \\ \diagdown \\ \phi_4 p_4 \end{array} = -2i(\lambda_1 + \lambda_2) \cos\left(\frac{p_1 \times p_2}{2}\right) \cos\left(\frac{p_3 \times p_4}{2}\right) - i\lambda_1 \cos\left(\frac{p_1 \times p_3}{2} + \frac{p_2 \times p_4}{2}\right) \quad (114)$$

$$\begin{array}{c} \sigma p_1 \\ \diagdown \\ \bullet \\ \diagup \\ \sigma p_2 \\ \diagup \\ \phi_4 p_3 \\ \diagdown \\ \phi_4 p_4 \end{array} = -2i(\lambda_1 + \lambda_2) \cos\left(\frac{p_1 \times p_2}{2}\right) \cos\left(\frac{p_3 \times p_4}{2}\right) - i\lambda_1 \cos\left(\frac{p_1 \times p_3}{2} + \frac{p_2 \times p_4}{2}\right) \quad (115)$$

$$\begin{array}{c} \sigma(\phi_4) p_1 \\ \diagdown \\ \bullet \\ \diagup \\ \sigma(\phi_4) p_2 \\ \diagup \\ \sigma(\phi_4) p_3 \\ \diagdown \\ \sigma(\phi_4) p_4 \end{array} = -2i\lambda \left[\cos\left(\frac{p_1 \times p_2}{2}\right) \cos\left(\frac{p_3 \times p_4}{2}\right) + \cos\left(\frac{p_1 \times p_3}{2}\right) \cos\left(\frac{p_2 \times p_4}{2}\right) + \cos\left(\frac{p_1 \times p_4}{2}\right) \cos\left(\frac{p_2 \times p_3}{2}\right) \right] \quad (116)$$

$$\begin{array}{c} \pi^* p_1 \\ \diagdown \\ \bullet \\ \diagup \\ \phi_4 p_2 \\ \diagup \\ \pi p_3 \\ \diagdown \\ \sigma p_4 \end{array} = -\lambda_1 \sin\left(\frac{p_1 \times p_2}{2} + \frac{p_3 \times p_4}{2}\right) \quad (117)$$

$$\begin{array}{c} \pi^* p_1 \\ \diagdown \\ \bullet \\ \diagup \\ \pi p_2 \\ \diagup \\ \sigma p_3 \end{array} = -2i\lambda a \cos\left(\frac{p_1 \times p_2}{2}\right) \quad (118)$$

$$\begin{array}{c} \sigma p_1 \\ \diagdown \\ \bullet \\ \diagup \\ \sigma p_2 \\ \diagup \\ \sigma p_3 \end{array} = -2i\lambda a \left[\cos\left(\frac{p_1 \times p_2}{2}\right) + \cos\left(\frac{p_1 \times p_3}{2}\right) + \cos\left(\frac{p_2 \times p_3}{2}\right) \right] \quad (119)$$

$$\begin{array}{c} \phi_4 \ p_1 \\ \vdots \\ \bullet \\ \text{---} \sigma \ p_3 \end{array} = -2i(\lambda + \lambda_1)a \cos\left(\frac{p_1 \times p_2}{2}\right) \quad (120)$$

$$\begin{array}{c} \phi_4 \ p_2 \\ \vdots \\ \pi \ p_1 \\ \vdots \\ \bullet \\ \text{---} \phi_4 \ p_3 \\ \vdots \\ \pi^* \ p_2 \end{array} = -\lambda_1 a \sin\left(\frac{p_1 \times p_2}{2}\right) \quad (121)$$

A.3 Matter Covariant Derivative Feynman Rules (Partial)

All momenta flow into the vertices.

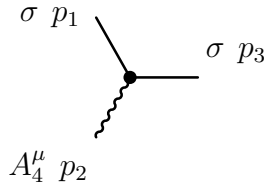
$$\begin{array}{c} \sigma \ p_1 \\ \text{---} \\ \bullet \\ \text{---} A^{*\mu} \ p_3 \end{array} = 2ia g^2 g_{\mu\nu} \cos\left(\frac{p_2 \times p_3}{2}\right) \quad (122)$$

$$\begin{array}{c} A^\nu \ p_2 \\ \text{---} \sigma \ p_1 \\ \text{---} \bullet \\ \text{---} \sigma \ p_2 \\ \text{---} A^\mu \ p_3 \quad \text{---} A^{*\nu} \ p_4 \end{array} = ig^2 g_{\mu\nu} \left[\cos\left(\frac{p_1 \times p_2}{2}\right) \cos\left(\frac{p_3 \times p_4}{2}\right) + \cos\left(\frac{p_1 \times p_4}{2} + \frac{p_2 \times p_3}{2}\right) \right] \quad (123)$$

$$\begin{array}{c} \sigma \ p_1 \\ \text{---} \\ \bullet \\ \text{---} \pi \ p_3 \end{array} = ig(p_3 - p_1)_\mu \cos\left(\frac{p_1 \times p_2}{2}\right) \quad (124)$$

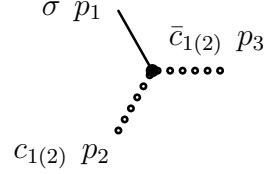
$$\begin{array}{c} \sigma \ p_1 \\ \text{---} \\ \bullet \\ \text{---} \pi^* \ p_3 \end{array} = ig(p_1 - p_3)_\mu \cos\left(\frac{p_1 \times p_2}{2}\right) \quad (125)$$

$$\begin{array}{c} \sigma \ p_1 \\ \text{---} \\ \bullet \\ \text{---} \phi_4 \ p_3 \\ \text{---} A_3^\mu \ p_2 \end{array} = g(p_3 - p_1)_\mu \sin\left(\frac{p_1 \times p_2}{2}\right) \quad (126)$$



$$= g(p_3 - p_1)_\mu \sin\left(\frac{p_1 \times p_2}{2}\right) \quad (127)$$

A.4 Ghost-Ghost-Higgs Interaction



$$= -\frac{i\xi M^2}{a} \cos\left(\frac{p_2 \times p_3}{2}\right) \quad (128)$$

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